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On double circular inclusion problem in antiplane piezoelectricity

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Abstract

In this paper, an analytical solution in series form for the problem of double circular piezoelectric inclusions embedded in an infinite piezoelectric matrix is presented within the framework of linear theory of piezoelectricity. The matrix is subjected to remote electro-mechanical loading, and the three phase system is also subjected to the action of arbitrary singularities. The solution is obtained by applying complex potential approach in conjunction with the techniques of conformal mapping, analytical continuation, singularity analysis, Laurent's series expansion in an annular ring region and Cauchy integral formulae, etc. Based on the obtained complex potentials, explicit expressions for the stress and electric displacement in the matrix and the two circular inclusions are also derived. A numerical investigation for the case of remote loading is performed to illustrate the influence of a third phase on the system's electroelastic coupling behavior and also to verify the correctness and usefulness of the solution. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Due to their well-known intrinsic electro-mechanical coupling phenomenon, piezoelectric materials are widely used in technology, such as high power sonar transducers, electro-mechanical actuators, and piezoelectric power supplies. These devices are designed to work under electro-mechanical loading conditions. The existence of macro as well as micro flaws and defects, such as dislocations, cracks and inclusions, etc. will seriously affect the lifetime of these devices.

Considerable research has been carried out on the behavior of piezoelectric materials in the presence of dislocations, cracks and inclusions. Deeg (1980) examined the effect of a single dislocation, a single crack and a single inclusion on the coupled response of piezoelectric materials. Pak (1990) derived closed-form solution for a screw dislocation in an infinite piezoelectric solid, and showed the effect of a dislocation on the coupling behavior. Meguid and Deng (1998) obtained the solution for the interaction problem of a

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dislocation outside an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. Deng and Meguid (1999) provided a general treatment to the electro-elastic interaction problem of a screw dislocation inside an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. All of the aforementioned work only considered a single inclusion in an infinite matrix. In reality, when there exist many inhomogeneities in a piezoelectric matrix and when the interaction effect among the closely spaced inhomogeneities cannot be ignored, study on multi-inclusion problem becomes a crucial and also very interesting research topic.

The problem of double circular piezoelectric inclusions embedded in an infinite piezoelectric matrix is considered in this paper. The matrix is subjected to remote inplane electric field as well as out of plane shear; the matrix and the two inclusions are also subjected to the action of singularities which can include a screw dislocation, electric potential discontinuity, a point force and a point charge. In this study, the complex potential approach combined with the techniques of conformal mapping, analytical continuation, singularity analysis, Laurent's expansion in an annular ring region and Cauchy integral formulae, etc. is utilized to obtain an analytical solution in series form for the two inclusion problem. It can be easily observed from the obtained solution that the existence of a third phase significantly altered the mechanical and electrical fields when there is only one inclusion in the matrix. One apparent example is that under remote electro-mechanical loading, the stress and electric displacement within the two inclusions are no longer uniform as in the case of a single inclusion.

2. Problem statement and basic formulation

Now, consider two circular piezoelectric inclusions whose radii are 1 and $(x_1 - x_2)/2$, respectively embedded in an infinite piezoelectric matrix. The two inclusions and the matrix have different elastic and electric properties and are assumed to be transversely isotropic with respect to the longitudinal direction. The matrix is subjected to remote in-plane electric field as well as out-of-plane shear; the matrix and the two inclusions are also subjected to the action of singularities which can be a screw dislocation, an electric potential discontinuity, a point force and a point charge. The two inhomogeneities are assumed to be perfectly bonded with the matrix along the two interfaces and there are no concentrated forces and free charges lying along the two interfaces. The region occupied by the matrix is denoted as region D_2, while the region occupied by the left inclusion is denoted as region D_3 and the region occupied by the right inclusion is denoted as region D_1. The subscripts for complex potentials also use the corresponding notations.

The governing field equations

$$\sigma_{zx,x} + \sigma_{zy,y} = 0, \qquad D_{x,x} + D_{y,y} = 0. \tag{1}$$

The constitutive equations

$$\begin{Bmatrix} \sigma_{zy} \\ D_y \end{Bmatrix} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix} \begin{Bmatrix} w_{,y} \\ \phi_{,y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} w_{,y} \\ \phi_{,y} \end{Bmatrix}, \tag{2a}$$

$$\begin{Bmatrix} \sigma_{zx} \\ D_x \end{Bmatrix} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix} \begin{Bmatrix} w_{,x} \\ \phi_{,x} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} w_{,x} \\ \phi_{,x} \end{Bmatrix}, \tag{2b}$$

where in Eqs. (1), (2a) and (2b), σ_{zx}, σ_{zy} are the shear stress components, D_x and D_y are the electric displacement components, w is the out-of-plane displacement, ϕ is the electric potential, the constitutive constants c_{44}, e_{15}, ε_{11} are, respectively, the longitudinal shear modulus, piezoelectric modulus and dielectric modulus. Substitution of Eqs. (2a) and (2b) into Eq. (1) yields

$$\boldsymbol{\Gamma}\begin{Bmatrix} \nabla^2 w \\ \nabla^2 \phi \end{Bmatrix} = \mathbf{0}. \tag{3}$$

The general solution to Eq. (3) is

$$\mathbf{U} = \begin{Bmatrix} w \\ \phi \end{Bmatrix} = \operatorname{Im}\mathbf{f}(z), \tag{4}$$

where in the above equation, $\mathbf{f}(z)$ is a two-dimensional analytical complex function vector of a complex variable $z = x + \mathrm{i}y$.

The mechanical strain and electric field strength can be obtained from Eq. (4) as follows:

$$\begin{Bmatrix} \gamma_{zy} + \mathrm{i}\gamma_{zx} \\ -E_y - \mathrm{i}E_x \end{Bmatrix} = \mathbf{f}'(z). \tag{5}$$

On substituting Eq. (4) into Eqs. (2a) and (2b), we can obtain

$$\begin{Bmatrix} \sigma_{zy} \\ D_y \end{Bmatrix} = \boldsymbol{\Gamma}\operatorname{Re}\mathbf{f}'(z), \tag{6a}$$

$$\begin{Bmatrix} \sigma_{zx} \\ D_x \end{Bmatrix} = \boldsymbol{\Gamma}\operatorname{Im}\mathbf{f}'(z). \tag{6b}$$

From Eqs. (6a) and (6b), we can get

$$\begin{Bmatrix} \sigma_{zy} + \mathrm{i}\sigma_{zx} \\ D_y + \mathrm{i}D_x \end{Bmatrix} = \boldsymbol{\Gamma}\mathbf{f}'(z). \tag{7}$$

Introduce a potential function vector $\boldsymbol{\Phi}$ which satisfies the following relationships:

$$\begin{Bmatrix} \sigma_{zy} \\ D_y \end{Bmatrix} = \boldsymbol{\Phi}_{,x}, \qquad \begin{Bmatrix} \sigma_{zx} \\ D_x \end{Bmatrix} = -\boldsymbol{\Phi}_{,y}. \tag{8}$$

From Eqs. (6a) and (6b), we can get

$$\boldsymbol{\Phi} = \boldsymbol{\Gamma}\operatorname{Re}\mathbf{f}(z). \tag{9}$$

From Eqs. (4) and (9), the boundary conditions of mechanical and electric fields are

$$\begin{cases} \mathbf{U} = \operatorname{Im}\mathbf{f}(z) \\ \boldsymbol{\Phi} = \boldsymbol{\Gamma}\operatorname{Re}\mathbf{f}(z). \end{cases} \tag{10}$$

Now, introduce the following form of fractional linear mapping function (Zhuang and Zhang, 1984):

$$z = \frac{\zeta - a}{a\zeta - 1}, \tag{11}$$

$$\text{where } a = \frac{1 + x_1x_2 + \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 + x_2}, \qquad R = \frac{x_1x_2 - 1 - \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 - x_2}.$$

The above mapping function is conformal in the whole plane including infinity, and it can map the two circular inclusions whose radii are 1 and $(x_1 - x_2)/2$ in the z-plane onto $|\zeta| > 1$ and $|\zeta| < R$ in the ζ-plane while the matrix region in the z-plane is mapped onto an annular ring $R < |\zeta| < 1$ in the ζ-plane, infinity in the z-plane is mapped to $\zeta = 1/a$ in the ζ-plane as shown in Fig. 1.

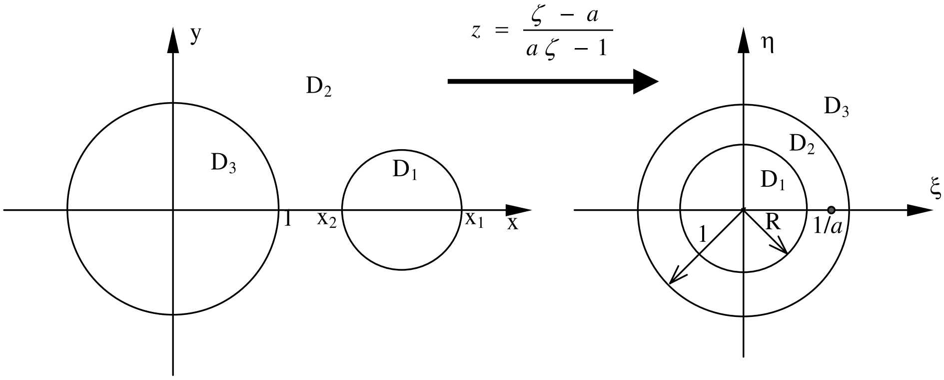

Fig. 1. A schematic representation of the conformal mapping adopted.

With the mapping function (11), Eq. (10) can be rewritten in the ζ-plane as follows:

$$\begin{cases} \mathbf{U} = \mathrm{Im}\,\mathbf{f}(\zeta) \\ \mathbf{\Phi} = \mathbf{\Gamma}\,\mathrm{Re}\,\mathbf{f}(\zeta). \end{cases} \tag{12}$$

In the z-plane, the singular behavior of $\mathbf{f}_1(z)$, $\mathbf{f}_2(z)$, $\mathbf{f}_3(z)$ including infinity is

$$\mathbf{f}_{1s}(z) = \frac{\hat{\mathbf{b}}_1 - \mathbf{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(z - z_1), \tag{13}$$

$$\mathbf{f}_{2s}(z) = \mathbf{\Pi}z + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln(z - z_2) + \frac{\hat{\mathbf{b}}_1 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(z - z_1) + \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(z - z_3), \tag{14}$$

$$\mathbf{f}_{3s}(z) = \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(z - z_3), \tag{15}$$

where in the above equation, the real quantity vector $\hat{\mathbf{b}}$ represents a screw dislocation and electric potential discontinuity, while the real quantity vector $\hat{\mathbf{f}}$ represents a point force and a point charge. $\mathbf{\Pi}$ is determined by the remote loading conditions, which have the following four possible combinations:

Case 1: remote mechanical strains γ_{zx}^{∞}, γ_{zy}^{∞} and remote electric field strength E_x^{∞}, E_y^{∞}

$$\mathbf{\Pi} = \begin{Bmatrix} \gamma_{zy}^{\infty} + \mathrm{i}\gamma_{zx}^{\infty} \\ -E_y^{\infty} - \mathrm{i}E_x^{\infty} \end{Bmatrix}. \tag{16a}$$

Case 2: remote mechanical stresses σ_{zx}^{∞}, σ_{zy}^{∞} and remote electric displacement D_x^{∞}, D_y^{∞}

$$\mathbf{\Pi} = \frac{1}{c_{44}^2\varepsilon_{11}^2 + (e_{15}^2)^2}\begin{bmatrix} \varepsilon_{11}^2 & e_{15}^2 \\ e_{15}^2 & -c_{44}^2 \end{bmatrix}\begin{Bmatrix} \sigma_{zy}^{\infty} + \mathrm{i}\sigma_{zx}^{\infty} \\ D_y^{\infty} + \mathrm{i}D_x^{\infty} \end{Bmatrix}. \tag{16b}$$

Case 3: remote mechanical strains γ_{zx}^{∞}, γ_{zy}^{∞} and remote electric displacement D_x^{∞}, D_y^{∞}

$$\mathbf{\Pi} = \frac{1}{\varepsilon_{11}^2}\begin{bmatrix} \varepsilon_{11}^2 & 0 \\ e_{15}^2 & -1 \end{bmatrix}\begin{Bmatrix} \gamma_{zy}^{\infty} + \mathrm{i}\gamma_{zx}^{\infty} \\ D_y^{\infty} + \mathrm{i}D_x^{\infty} \end{Bmatrix}. \tag{16c}$$

Case 4: remote mechanical stresses σ_{zx}^{∞}, σ_{zy}^{∞} and remote electric field strength E_x^{∞}, E_y^{∞}

$$\mathbf{\Pi} = \frac{1}{c_{44}^2}\begin{bmatrix} 1 & -e_{15}^2 \\ 0 & c_{44}^2 \end{bmatrix}\begin{Bmatrix} \sigma_{zy}^{\infty} + \mathrm{i}\sigma_{zx}^{\infty} \\ -E_y^{\infty} - \mathrm{i}E_x^{\infty} \end{Bmatrix}. \tag{16d}$$

Then, in the ζ-plane, $\mathbf{f}_1(\zeta)$, $\mathbf{f}_2(\zeta)$, $\mathbf{f}_3(\zeta)$ will possess the following singularities in their respective definition regions:

$$\mathbf{f}_{1s}(\zeta) = \frac{\hat{\mathbf{b}}_1 - \mathbf{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - \zeta_1), \tag{17}$$

$$\mathbf{f}_{2s}(\zeta) = \frac{\mathbf{K}}{\zeta - 1/a} + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_2}{\zeta - 1/a} - \frac{\hat{\mathbf{b}}_1 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/a}{\zeta} - \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a), \tag{18}$$

$$\mathbf{f}_{3s}(\zeta) = \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_3}{\zeta}, \tag{19}$$

where in Eq. (18), $\mathbf{K} = (1/a^2 - 1)\mathbf{\Pi}$.

The complex potentials defined in the two inclusions and the matrix must satisfy their respective singularity conditions (17)–(19) and the continuity conditions across the two interfaces $|t| = R$ and $|t| = 1$. The continuity conditions can be specifically expressed as follows:

$$\mathbf{U}_1 = \mathbf{U}_2, \qquad \mathbf{\Phi}_1 = \mathbf{\Phi}_2 \quad (|t| = R), \tag{20}$$

$$\mathbf{U}_2 = \mathbf{U}_3, \qquad \mathbf{\Phi}_2 = \mathbf{\Phi}_3 \quad (|t| = 1). \tag{21}$$

The above continuity conditions can also be expressed in terms of complex potentials as

$$\begin{cases} \mathbf{f}_1(t) - \overline{\mathbf{f}_1(t)} = \mathbf{f}_2(t) - \overline{\mathbf{f}_2(t)}, \\ \mathbf{\Gamma}_1(\mathbf{f}_1(t) + \overline{\mathbf{f}_1(t)}) = \mathbf{\Gamma}_2(\mathbf{f}_2(t) + \overline{\mathbf{f}_2(t)}), \end{cases} \quad |t| = R, \tag{22}$$

$$\begin{cases} \mathbf{f}_2(t) - \overline{\mathbf{f}_2(t)} = \mathbf{f}_3(t) - \overline{\mathbf{f}_3(t)}, \\ \mathbf{\Gamma}_2(\mathbf{f}_2(t) + \overline{\mathbf{f}_2(t)}) = \mathbf{\Gamma}_3(\mathbf{f}_3(t) + \overline{\mathbf{f}_3(t)}), \end{cases} \quad |t| = 1. \tag{23}$$

3. General solution

The main idea of the solution procedure is the following:

First, we satisfy all of the continuity conditions across the interface $|t| = R$ to obtain expressions of $\mathbf{f}_1(\zeta)$ and $\mathbf{f}_2(\zeta)$.

Second, we satisfy all of the continuity conditions across the interface $|t| = 1$ to obtain expressions of $\mathbf{f}_2(\zeta)$ and $\mathbf{f}_3(\zeta)$.

Third, by equating the two expressions of $\mathbf{f}_2(\zeta)$ obtained from satisfying the continuity conditions across $|t| = R$ and $|t| = 1$, the unknown coefficients in the expressions can be uniquely determined.

Fourth, based on the obtained complex potentials, physical quantities such as stresses and electric displacement can be derived.

Based on the superposition principle for a linear system, the original problem can be decomposed into the following three cases which can be treated more easily: 1. Remote mechanical and electrical loadings, 2. a singularity in the matrix and 3. one singularity in each of the two inclusions. The solutions for the above three cases will be derived separately in the following analysis.

3.1. Remote uniform mechanical and electrical loadings

First, the continuity conditions across inner circle $|t| = R$ should be satisfied.

Introducing the following forms of analytical continuation

$$\mathbf{f}_1(\zeta) = \overline{\mathbf{f}}_1(R^2/\zeta) \quad (|\zeta| > R), \tag{24}$$

$$\mathbf{f}_2(\zeta) = \overline{\mathbf{f}}_2(R^2/\zeta) \quad (R^2 < |\zeta| < R), \tag{25}$$

then, the continuity of displacement and electric potential can be expressed as follows:

$$\mathbf{f}_1^+ - \mathbf{f}_1^- = \mathbf{f}_2^- - \mathbf{f}_2^+, \tag{26}$$

where superscript + denotes approaching $|t| = R$ from within the circle, superscript − denotes approaching $|t| = R$ from outside the circle.

Eq. (26) can be rearranged as follows:

$$(\mathbf{f}_1 + \mathbf{f}_2)^+ - (\mathbf{f}_1 + \mathbf{f}_2)^- = \mathbf{0}. \tag{27}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_2(\zeta)$, the following expression can be obtained:

$$\mathbf{G}_1(\zeta) = \mathbf{f}_1(\zeta) + \mathbf{f}_2(\zeta) = \mathbf{A}_0 + \sum_{n=1}^{+\infty}(\mathbf{A}_n\zeta^n + R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}) + \frac{\mathbf{K}}{\zeta - 1/a} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2} \quad (R^2 < |\zeta| < 1). \tag{28}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\boldsymbol{\Gamma}_1(\mathbf{f}_1^+ + \mathbf{f}_1^-) = \boldsymbol{\Gamma}_2(\mathbf{f}_2^- + \mathbf{f}_2^+). \tag{29}$$

Substituting Eq. (28) into Eq. (29) yields

$$\mathbf{f}_1^+ + \mathbf{f}_1^- = 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\mathbf{G}_1(t). \tag{30}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_1(\zeta) = \begin{cases} 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\left(\mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n + \frac{\mathbf{K}}{\zeta - 1/a}\right) & |\zeta| < R \\ 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2}\right) & |\zeta| > R. \end{cases} \tag{31}$$

Substituting Eq. (31) into Eq. (28), $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ can be expressed as follows:

$$\mathbf{f}_2(\zeta) = \mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n + (\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2}\right) + \frac{\mathbf{K}}{\zeta - 1/a}. \tag{32}$$

Second, the continuity conditions across outer circle $|t| = 1$ should be satisfied.

Introducing the following forms of analytical continuation

$$\mathbf{f}_3(\zeta) = \overline{\mathbf{f}}_3(1/\zeta) \quad (|\zeta| < 1), \tag{33}$$

$$\mathbf{f}_2(\zeta) = \overline{\mathbf{f}}_2(1/\zeta) \quad (1 < |\zeta| < 1/R), \tag{34}$$

then, the continuity of displacement and electric potential can be expressed as follows:

$$\mathbf{f}_2^+ - \mathbf{f}_2^- = \mathbf{f}_3^- - \mathbf{f}_3^+, \tag{35}$$

where superscript + denotes approaching $|t| = 1$ from within the circle, superscript − denotes approaching $|t| = 1$ from outside the circle. Eq. (35) can be rearranged as

$$(\mathbf{f}_2 + \mathbf{f}_3)^+ - (\mathbf{f}_2 + \mathbf{f}_3)^- = \mathbf{0}. \tag{36}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_2(\zeta)$, the following expression can be obtained:

$$\mathbf{G}_2(\zeta) = \mathbf{f}_2(\zeta) + \mathbf{f}_3(\zeta) = \mathbf{B}_0 + \sum_{n=1}^{+\infty}(\mathbf{B}_n\zeta^n + \overline{\mathbf{B}}_n\zeta^{-n}) + \frac{\mathbf{K}}{\zeta - 1/a} - \frac{a^2\overline{\mathbf{K}}}{\zeta - a} \quad (R < |\zeta| < 1/R). \tag{37}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\boldsymbol{\Gamma}_2(\mathbf{f}_2^+ + \mathbf{f}_2^-) = \boldsymbol{\Gamma}_3(\mathbf{f}_3^- + \mathbf{f}_3^+). \tag{38}$$

Substituting Eq. (37) into Eq. (38) yields

$$\mathbf{f}_3^+ + \mathbf{f}_3^- = 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\mathbf{G}_2(t). \tag{39}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_3(\zeta) = \begin{cases} 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left(\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{a^2\overline{\mathbf{K}}}{\zeta - a}\right), & |\zeta| < 1, \\ 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} + \frac{\mathbf{K}}{\zeta - 1/a}\right), & |\zeta| > 1. \end{cases} \tag{40}$$

Substituting Eq. (40) into Eq. (37), $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ can be expressed as follows:

$$\mathbf{f}_2(\zeta) = \sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} + (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left(\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{a^2\overline{\mathbf{K}}}{\zeta - a}\right) + \frac{\mathbf{K}}{\zeta - 1/a}. \tag{41}$$

The two expressions of $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ obtained from satisfying the continuity conditions across the two interfaces must be the same one, then equating Eqs. (32) and (41) yields

$$\begin{aligned} &\mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n + (\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2}\right) \\ &\quad = \sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} + (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left(\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{a^2\overline{\mathbf{K}}}{\zeta - a}\right). \end{aligned} \tag{42}$$

The above equation can be separated into the following two equations:

$$\mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n = (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left(\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{a^2\overline{\mathbf{K}}}{\zeta - a}\right), \tag{43}$$

$$\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} = (\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2}\right). \tag{44}$$

When $R < |\zeta| < 1$, the following power series expansion is convergent:

$$\frac{a^2}{\zeta - a} = -\sum_{n=0}^{+\infty}a^{-(n-1)}\zeta^n, \tag{45}$$

$$\frac{a^2R^2}{\zeta - aR^2} = \sum_{n=1}^{+\infty} a(aR^2)^n \zeta^{-n}. \tag{46}$$

Substituting Eqs. (45) and (46) into Eqs. (43) and (44), comparing the coefficients of the same power of ζ, we can obtain

$$\begin{bmatrix} \mathbf{\Gamma}_2 + \mathbf{\Gamma}_3 & \mathbf{\Gamma}_2 - \mathbf{\Gamma}_3 \\ (\mathbf{\Gamma}_2 - \mathbf{\Gamma}_1)R^{2n} & \mathbf{\Gamma}_2 + \mathbf{\Gamma}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \end{Bmatrix} = \begin{Bmatrix} -(\mathbf{\Gamma}_2 - \mathbf{\Gamma}_3)a^{-(n-1)}\overline{\mathbf{K}} \\ (\mathbf{\Gamma}_2 - \mathbf{\Gamma}_1)a^{n+1}R^{2n}\mathbf{K} \end{Bmatrix} \quad (n = 1, 2, \ldots, +\infty). \tag{47}$$

From the above equation, we can get those unknown constants as follows

$$\begin{Bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \end{Bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_2 + \mathbf{\Gamma}_3 & \mathbf{\Gamma}_2 - \mathbf{\Gamma}_3 \\ (\mathbf{\Gamma}_2 - \mathbf{\Gamma}_1)R^{2n} & \mathbf{\Gamma}_2 + \mathbf{\Gamma}_1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0}_{2\times 2} & -(\mathbf{\Gamma}_2 - \mathbf{\Gamma}_3)a^{-(n-1)} \\ (\mathbf{\Gamma}_2 - \mathbf{\Gamma}_1)a^{n+1}R^{2n} & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{Bmatrix} \mathbf{K} \\ \overline{\mathbf{K}} \end{Bmatrix},$$
$$n = 1, 2, \ldots, +\infty. \tag{48}$$

Since $\mathbf{A}_0$ and $\mathbf{B}_0$ represent the equipotential field and the translation of a rigid body, they can be ignored. The complex potentials which characterize mechanical and electrical fields have been obtained completely as follows:

$$\mathbf{f}_1(\zeta) = 2(\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2 \left(\sum_{n=1}^{+\infty} \mathbf{A}_n \zeta^n + \frac{\mathbf{K}}{\zeta - 1/a} \right), \quad |\zeta| < R,$$

$$\begin{aligned} \mathbf{f}_2(\zeta) &= \sum_{n=1}^{+\infty} \mathbf{A}_n \zeta^n + (\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2) \left(\sum_{n=1}^{+\infty} R^{2n}\overline{\mathbf{A}}_n \zeta^{-n} - \frac{a^2R^2\overline{\mathbf{K}}}{\zeta - aR^2} \right) + \frac{\mathbf{K}}{\zeta - 1/a} \\ &= \sum_{n=1}^{+\infty} \overline{\mathbf{B}}_n \zeta^{-n} + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2) \left(\sum_{n=1}^{+\infty} \mathbf{B}_n \zeta^n - \frac{a^2\overline{\mathbf{K}}}{\zeta - a} \right) + \frac{\mathbf{K}}{\zeta - 1/a}, \quad R < |\zeta| < 1, \end{aligned}$$

$$\mathbf{f}_3(\zeta) = 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2 \left(\sum_{n=1}^{+\infty} \overline{\mathbf{B}}_n \zeta^{-n} + \frac{\mathbf{K}}{\zeta - 1/a} \right), \quad |\zeta| > 1.$$

The expressions of stress and electric displacement within the two inclusions and the matrix can be derived expediently from the above complex potentials. The distribution of stress and electric displacement in the right inclusion is

$$\begin{Bmatrix} \sigma_{zy} + \mathrm{i}\sigma_{zx} \\ D_y + \mathrm{i}D_x \end{Bmatrix} = 2\mathbf{\Gamma}_1(\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2 \left(\sum_{n=1}^{+\infty} n \frac{(a\zeta - 1)^2 \zeta^{n-1}}{a^2 - 1} \mathbf{A}_n - \frac{a^2}{a^2 - 1}\mathbf{K} \right), \quad |\zeta| < R. \tag{49}$$

The distribution of stress and electric displacement in the left inclusion is

$$\begin{Bmatrix} \sigma_{zy} + \mathrm{i}\sigma_{zx} \\ D_y + \mathrm{i}D_x \end{Bmatrix} = 2\mathbf{\Gamma}_3(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2 \left(\sum_{n=1}^{+\infty} - n \frac{(a\zeta - 1)^2 \zeta^{-(n+1)}}{a^2 - 1} \overline{\mathbf{B}}_n - \frac{a^2}{a^2 - 1}\mathbf{K} \right), \quad |\zeta| > 1. \tag{50}$$

From the above two expressions, it is not difficult to observe that the stresses and electric displacement within the two inclusions are no longer uniform due to the interactions of the inclusions.

The distribution of stress and electric displacement in the matrix is

$$\begin{Bmatrix} \sigma_{zy} + \mathrm{i}\sigma_{zx} \\ D_y + \mathrm{i}D_x \end{Bmatrix} = \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty} - nR^{2n}\frac{(a\zeta - 1)^2\zeta^{-(n+1)}}{a^2 - 1}\overline{\mathbf{A}}_n + \frac{a^2R^2}{a^2 - 1}\frac{(a\zeta - 1)^2}{(\zeta - aR^2)^2}\overline{\mathbf{K}} \right)$$
$$+ \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} n\frac{(a\zeta - 1)^2\zeta^{n-1}}{a^2 - 1}\mathbf{A}_n - \frac{a^2}{a^2 - 1}\mathbf{K} \right), \quad R < |\zeta| < 1$$
$$= \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty} n\frac{(a\zeta - 1)^2\zeta^{n-1}}{a^2 - 1}\mathbf{B}_n + \frac{a^2}{a^2 - 1}\frac{(a\zeta - 1)^2}{(\zeta - a)^2}\overline{\mathbf{K}} \right)$$
$$+ \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} - n\frac{(a\zeta - 1)^2\zeta^{-(n+1)}}{a^2 - 1}\overline{\mathbf{B}}_n - \frac{a^2}{a^2 - 1}\mathbf{K} \right), \quad R < |\zeta| < 1. \tag{51}$$

The stress and electric displacement along the boundary $|t| = R$ are

$$\begin{Bmatrix} \sigma_{zt} + \mathrm{i}\sigma_{zn} \\ D_t + \mathrm{i}D_n \end{Bmatrix}^+ = 2\boldsymbol{\Gamma}_1(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} nt^n\mathbf{A}_n - \frac{a^2t}{(at - 1)^2}\mathbf{K} \right)\frac{|at - 1|^2}{R(a^2 - 1)}, \tag{52}$$

$$\begin{Bmatrix} \sigma_{zt} + \mathrm{i}\sigma_{zn} \\ D_t + \mathrm{i}D_n \end{Bmatrix}^- = \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} nt^n\mathbf{A}_n - \frac{a^2t}{(at - 1)^2}\mathbf{K} \right)\frac{|at - 1|^2}{R(a^2 - 1)} + \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)$$
$$\times \left(\sum_{n=1}^{+\infty} -nR^{2n}t^{-n}\overline{\mathbf{A}}_n + \frac{a^2R^2t}{(t - aR^2)^2}\overline{\mathbf{K}} \right)\frac{|at - 1|^2}{R(a^2 - 1)}. \tag{53}$$

The stress and electric displacement along the boundary $|t| = 1$ are

$$\begin{Bmatrix} \sigma_{zt} + \mathrm{i}\sigma_{zn} \\ D_t + \mathrm{i}D_n \end{Bmatrix}^+ = \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} - nt^{-n}\overline{\mathbf{B}}_n - \frac{a^2t}{(at - 1)^2}\mathbf{K} \right)\frac{|at - 1|^2}{1 - a^2} + \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)$$
$$\times \left(\sum_{n=1}^{+\infty} n\mathbf{B}_nt^n + \frac{a^2t}{(t - a)^2}\overline{\mathbf{K}} \right)\frac{|at - 1|^2}{1 - a^2}, \tag{54}$$

$$\begin{Bmatrix} \sigma_{zt} + \mathrm{i}\sigma_{zn} \\ D_t + \mathrm{i}D_n \end{Bmatrix}^- = 2\boldsymbol{\Gamma}_3(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} - nt^{-n}\overline{\mathbf{B}}_n - \frac{a^2t}{(at - 1)^2}\mathbf{K} \right)\frac{|at - 1|^2}{1 - a^2}, \tag{55}$$

where in Eqs. (52)–(55), subscript t stands for tangential component, whereas the subscript n stands for normal component.

3.2. A singularity in the matrix

First, the continuity conditions across inner circle $|t| = R$ should be satisfied.

Introducing the analytical continuation expressed by Eqs. (24) and (25), then the continuity of displacement and electric potential can be expressed as follows:

$$(\mathbf{f}_1 + \mathbf{f}_2)^+ - (\mathbf{f}_1 + \mathbf{f}_2)^- = \mathbf{0}. \tag{56}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_2(\zeta)$, the following expression can be obtained:

$$\begin{aligned}\mathbf{G}_1(\zeta) &= \mathbf{f}_1(\zeta) + \mathbf{f}_2(\zeta) \\ &= \mathbf{A}_0 + \sum_{n=1}^{+\infty}(\mathbf{A}_n\zeta^n + R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}) + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_2}{\zeta - 1/a} \\ &\quad + \frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - R^2/\overline{\zeta}_2}{\zeta - aR^2} \quad (R^2 < |\zeta| < 1). \end{aligned} \tag{57}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\mathbf{\Gamma}_1(\mathbf{f}_1^+ + \mathbf{f}_1^-) = \mathbf{\Gamma}_2(\mathbf{f}_2^- + \mathbf{f}_2^+). \tag{58}$$

Substituting Eq. (57) into Eq. (58) yields:

$$\mathbf{f}_1^+ + \mathbf{f}_1^- = 2(\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\mathbf{G}_1(t). \tag{59}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_1(\zeta) = \begin{cases} 2(\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\left(\mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}\right), & |\zeta| < R, \\ 2(\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} + \frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-R^2/\overline{\zeta_2}}{\zeta-aR^2}\right), & |\zeta| > R. \end{cases} \tag{60}$$

Substituting Eq. (60) into Eq. (57), $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ can be expressed as

$$\begin{aligned}\mathbf{f}_2(\zeta) &= \mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_2}{\zeta - 1/a} + (\mathbf{\Gamma}_1 + \mathbf{\Gamma}_2)^{-1}(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}\right. \\ &\quad \left. + \frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - R^2/\overline{\zeta}_2}{\zeta - aR^2}\right). \end{aligned} \tag{61}$$

Second, the continuity conditions across outer circle $|t| = 1$ should be satisfied.

Introducing the analytical continuation expressed by Eqs. (33) and (34), then the continuity of displacement and electric potential can be expressed as follows:

$$(\mathbf{f}_2 + \mathbf{f}_3)^+ - (\mathbf{f}_2 + \mathbf{f}_3)^- = \mathbf{0}. \tag{62}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_2(\zeta)$, the following expression can be obtained:

$$\begin{aligned}\mathbf{G}_2(\zeta) &= \mathbf{f}_2(\zeta) + \mathbf{f}_3(\zeta) \\ &= \mathbf{B}_0 + \sum_{n=1}^{+\infty}(\mathbf{B}_n\zeta^n + \overline{\mathbf{B}}_n\zeta^{-n}) + \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_2}{\zeta - 1/a} \\ &\quad + \frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/\overline{\zeta_2}}{\zeta - a} \quad (R < |\zeta| < 1/R). \end{aligned} \tag{63}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\mathbf{\Gamma}_2(\mathbf{f}_2^+ + \mathbf{f}_2^-) = \mathbf{\Gamma}_3(\mathbf{f}_3^- + \mathbf{f}_3^+). \tag{64}$$

Substituting Eq. (63) into Eq. (64) yields

$$\mathbf{f}_3^+ + \mathbf{f}_3^- = 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2\mathbf{G}_2(t). \tag{65}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_3(\zeta)=\begin{cases}2(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left(\mathbf{B}_0+\sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}\right), & |\zeta|<1,\\ 2(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}\right), & |\zeta|>1.\end{cases} \tag{66}$$

Substituting Eq. (66) into Eq. (63), $\mathbf{f}_2(\zeta)$ in its definition region, $R<|\zeta|<1$ can be expressed as

$$\mathbf{f}_2(\zeta)=\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}+(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}_2)\left(\mathbf{B}_0+\sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}\right). \tag{67}$$

The two expressions of $\mathbf{f}_2(\zeta)$ in its definition region $R<|\zeta|<1$ obtained from satisfying the continuity conditions across the two interfaces must be the same one, then equating Eqs. (61) and (67) yields

$$\begin{aligned}&\mathbf{A}_0+\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n+(\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-R^2/\overline{\zeta}_2}{\zeta-aR^2}\right)\\&\quad=\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}+(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}_2)\left(\mathbf{B}_0+\sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}\right).\end{aligned} \tag{68}$$

The above equation can be separated into the following two equations:

$$\mathbf{A}_0+\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n=(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}_2)\left(\mathbf{B}_0+\sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}\right), \tag{69}$$

$$\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}=(\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-R^2/\overline{\zeta}_2}{\zeta-aR^2}\right). \tag{70}$$

When $R<|\zeta|<1$, the following power series expansion is convergent:

$$\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}=\sum_{n=1}^{+\infty}\frac{a^{-n}-\overline{\zeta}_2^n}{n}\zeta^n, \tag{71}$$

$$\ln\frac{\zeta-R^2/\overline{\zeta}_2}{\zeta-aR^2}=\sum_{n=1}^{+\infty}\frac{(aR^2)^n-(R^2/\overline{\zeta}_2)^n}{n}\zeta^{-n}. \tag{72}$$

Substituting Eqs. (71) and (72) into Eqs. (69) and (70), then comparing the coefficients of the same power of ζ, we can obtain

$$\begin{bmatrix}\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3 & \boldsymbol{\Gamma}_2-\boldsymbol{\Gamma}_3\\ (\boldsymbol{\Gamma}_2-\boldsymbol{\Gamma}_1)R^{2n} & \boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_1\end{bmatrix}\begin{Bmatrix}\mathbf{A}_n\\ \mathbf{B}_n\end{Bmatrix}=\begin{Bmatrix}8 \quad (\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}_2)\frac{a^{-n}-\overline{\zeta}_2^n}{n}\frac{\hat{\mathbf{b}}_2+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\\ (\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)\frac{(aR^2)^n-(R^2/\zeta_2)^n}{n}\frac{\hat{\mathbf{b}}_2-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\end{Bmatrix}\quad(n=1,2,\ldots,+\infty). \tag{73}$$

From the above equation, we can get

$$\left\{\begin{array}{c}\mathbf{A}_n\\ \mathbf{B}_n\end{array}\right\}=\left[\begin{array}{cc}\mathbf{\Gamma}_2+\mathbf{\Gamma}_3 & \mathbf{\Gamma}_2-\mathbf{\Gamma}_3\\ (\mathbf{\Gamma}_2-\mathbf{\Gamma}_1)R^{2n} & \mathbf{\Gamma}_2+\mathbf{\Gamma}_1\end{array}\right]^{-1}\left[\begin{array}{cc}\mathbf{0}_{2\times 2} & (\mathbf{\Gamma}_3-\mathbf{\Gamma}_2)\frac{a^{-n}-\overline{\zeta}_2^n}{n}\\ (\mathbf{\Gamma}_1-\mathbf{\Gamma}_2)\frac{(aR^2)^n-(R^2/\zeta_2)^n}{n} & \mathbf{0}_{2\times 2}\end{array}\right]$$
$$\times\left\{\begin{array}{c}\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\\ \frac{\hat{\mathbf{b}}_2+\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\end{array}\right\},\quad (n=1,2,\ldots,+\infty). \tag{74}$$

Since $\mathbf{A}_0$ and $\mathbf{B}_0$ represent the equipotential field and the translation of a rigid body, they can be ignored. The complex potentials which characterize mechanical and electrical fields have been obtained completely as follows:

$$\mathbf{f}_1(\zeta)=2(\mathbf{\Gamma}_1+\mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\left(\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}\right),\quad |\zeta|<R,$$

$$\begin{aligned}\mathbf{f}_2(\zeta)&=\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}+(\mathbf{\Gamma}_1+\mathbf{\Gamma}_2)^{-1}(\mathbf{\Gamma}_1-\mathbf{\Gamma}_2)\\&\quad\times\left(\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2+\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-R^2\overline{\zeta}_2}{\zeta-aR^2}\right)\\&=\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}+(\mathbf{\Gamma}_2+\mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3-\mathbf{\Gamma}_2)\\&\quad\times\left(\sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n+\frac{\hat{\mathbf{b}}_2+\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-1/\overline{\zeta}_2}{\zeta-a}\right),\quad R<|\zeta|<1,\end{aligned}$$

$$\mathbf{f}_3(\zeta)=2(\mathbf{\Gamma}_2+\mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2\left(\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n}+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{\zeta-\zeta_2}{\zeta-1/a}\right),\quad |\zeta|>1.$$

The expressions of stress and electric displacement within the two inclusions and the matrix can be derived expediently from the above complex potentials.

The distribution of stress and electric displacement in the right inclusion is

$$\left\{\begin{array}{c}\sigma_{zy}+\mathrm{i}\sigma_{zx}\\ D_y+\mathrm{i}D_x\end{array}\right\}=2\mathbf{\Gamma}_1(\mathbf{\Gamma}_1+\mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\left(\sum_{n=1}^{+\infty}n\frac{(a\zeta-1)^2\zeta^{n-1}}{a^2-1}\mathbf{A}_n+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{a\zeta_2-1}{a^2-1}\frac{a\zeta-1}{\zeta-\zeta_2}\right),\quad |\zeta|<R. \tag{75}$$

The distribution of stress and electric displacement in the left inclusion is

$$\begin{aligned}\left\{\begin{array}{c}\sigma_{zy}+\mathrm{i}\sigma_{zx}\\ D_y+\mathrm{i}D_x\end{array}\right\}&=2\mathbf{\Gamma}_3(\mathbf{\Gamma}_2+\mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2\left(\sum_{n=1}^{+\infty}-n\frac{(a\zeta-1)^2\zeta^{-(n+1)}}{a^2-1}\overline{\mathbf{B}}_n\right.\\&\quad\left.+\frac{\hat{\mathbf{b}}_2-\mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{a\zeta_2-1}{a^2-1}\frac{a\zeta-1}{\zeta-\zeta_2}\right),\quad |\zeta|>1.\end{aligned} \tag{76}$$

The distribution of stress and electric displacement in the matrix is

$$\begin{Bmatrix} \sigma_{zy} + \mathrm{i}\sigma_{zx} \\ D_y + \mathrm{i}D_x \end{Bmatrix} = \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} n\frac{(a\zeta-1)^2\zeta^{n-1}}{a^2-1}\mathbf{A}_n + \frac{\hat{\mathbf{b}}_2 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{a\zeta_2-1}{a^2-1}\frac{a\zeta-1}{\zeta-\zeta_2}\right) + \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}$$

$$\times(\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)\left\{ \sum_{n=1}^{+\infty}(-n)\frac{(a\zeta-1)^2\zeta^{-(n+1)}}{a^2-1}R^{2n}\overline{\mathbf{A}}_n + \frac{\hat{\mathbf{b}}_2 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{R^2/\overline{\zeta}_2 - aR^2}{a^2-1}\frac{(a\zeta-1)^2}{(\zeta-R^2/\overline{\zeta}_2)(\zeta-aR^2)}\right\}$$

$$= \boldsymbol{\Gamma}_2\left(\sum_{n=1}^{+\infty} -n\frac{(a\zeta-1)^2\zeta^{-(n+1)}}{a^2-1}\overline{\mathbf{B}}_n + \frac{\hat{\mathbf{b}}_2 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{a\zeta_2-1}{a^2-1}\frac{a\zeta-1}{\zeta-\zeta_2}\right) + \boldsymbol{\Gamma}_2(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}$$

$$\times(\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}_2)\left\{ \sum_{n=1}^{+\infty} n\frac{(a\zeta-1)^2\zeta^{n-1}}{a^2-1}\mathbf{B}_n + \frac{\hat{\mathbf{b}}_2 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\frac{1/\overline{\zeta}_2 - a}{a^2-1}\frac{(a\zeta-1)^2}{(\zeta-1/\overline{\zeta}_2)(\zeta-a)}\right\}, \quad R<|\zeta|<1. \tag{77}$$

3.3. One singularity in each of the two inclusions

First, the continuity conditions across inner circle $|t| = R$ should be satisfied.

Introducing the analytical continuation expressed by Eqs. (24) and (25), then the continuity of displacement and electric potential can be expressed as follows:

$$(\mathbf{f}_1 + \mathbf{f}_2)^+ - (\mathbf{f}_1 + \mathbf{f}_2)^- = 0. \tag{78}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_1(\zeta)$ and $\mathbf{f}_2(\zeta)$, the following expression can be obtained:

$$\begin{aligned}\mathbf{G}_1(\zeta) &= \mathbf{f}_1(\zeta) + \mathbf{f}_2(\zeta) \\ &= \mathbf{A}_0 + \sum_{n=1}^{+\infty}(\mathbf{A}_n\zeta^n + R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}) - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta-1/a) - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{\zeta-aR^2}{\zeta} \\ &\quad + \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-\zeta_1) + \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-R^2/\overline{\zeta}_1) - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-1/a) \\ &\quad - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-aR^2), \quad (R^2<|\zeta|<1).\end{aligned} \tag{79}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\boldsymbol{\Gamma}_1(\mathbf{f}_1^+ + \mathbf{f}_1^-) = \boldsymbol{\Gamma}_2(\mathbf{f}_2^- + \mathbf{f}_2^+). \tag{80}$$

Substituting Eq. (79) into Eq. (80) yields

$$\mathbf{f}_1^+ + \mathbf{f}_1^- = 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\mathbf{G}_1(t). \tag{81}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_1(\zeta) = 2(\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\left\{\mathbf{A}_0+\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n-\frac{\hat{\mathbf{b}}_3-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta-1/a)+\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-R^2/\overline{\zeta}_1)-\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-1/a)\right\}+\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-\zeta_1)-\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\times\ln(\zeta-R^2/\overline{\zeta}_1),\quad |\zeta|<R, \tag{82a}$$

$$\mathbf{f}_1(\zeta) = 2(\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2\left\{\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}-\frac{\hat{\mathbf{b}}_3+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{\zeta-aR^2}{\zeta}+\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1 i}{2\pi}\ln(\zeta-\zeta_1)-\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-aR^2)\right\}-\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-\zeta_1)+\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\times\ln(\zeta-R^2/\overline{\zeta}_1),\quad |\zeta|>R. \tag{82b}$$

Substituting Eq. (82b) into Eq. (79), $\mathbf{f}_2(\zeta)$ in its definition region $R<|\zeta|<1$ can be expressed as

$$\mathbf{f}_2(\zeta) = (\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)\left\{\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n}-\frac{\hat{\mathbf{b}}_3+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{\zeta-aR^2}{\zeta}+\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-\zeta_1)-\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-aR^2)\right\}+\mathbf{A}_0+\sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n-\frac{\hat{\mathbf{b}}_3-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta-1/a)-\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\times\ln(\zeta-1/a)+\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-\zeta_1). \tag{83}$$

Second, the continuity conditions across the outer circle $|t|=1$ should be satisfied.

Introducing the analytical continuation expressed by Eqs. (33) and (34), then the continuity of displacement and electric potential can be expressed as follows:

$$(\mathbf{f}_2+\mathbf{f}_3)^+-(\mathbf{f}_2+\mathbf{f}_3)^-=\mathbf{0}. \tag{84}$$

By the generalized Liouville's theorem and the symmetric condition and also noting the singular behavior of $\mathbf{f}_2(\zeta)$ and $\mathbf{f}_3(\zeta)$, the following expression can be obtained:

$$\begin{aligned}\mathbf{G}_2(\zeta)&=\mathbf{f}_2(\zeta)+\mathbf{f}_3(\zeta)\\&=\mathbf{B}_0+\sum_{n=1}^{+\infty}(\mathbf{B}_n\zeta^n+\overline{\mathbf{B}}_n\zeta^{-n})-\frac{\hat{\mathbf{b}}_1-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta-1/a}{\zeta}-\frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta-a)+\frac{\hat{\mathbf{b}}_3-\boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\\&\quad\times\ln(\zeta-\zeta_3)+\frac{\hat{\mathbf{b}}_3+\boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta-1/\overline{\zeta}_3)-\frac{\hat{\mathbf{b}}_3-\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta-1/a)-\frac{\hat{\mathbf{b}}_3+\boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\\&\quad\times\ln(\zeta-a)\quad(R<|\zeta|<1/R).\end{aligned} \tag{85}$$

The continuity of traction and normal component of electric displacement can be expressed as follows:

$$\boldsymbol{\Gamma}_2(\mathbf{f}_2^++\mathbf{f}_2^-)=\boldsymbol{\Gamma}_3(\mathbf{f}_3^-+\mathbf{f}_3^+). \tag{86}$$

Substituting Eq. (85) into Eq. (86) yields

$$\mathbf{f}_3^++\mathbf{f}_3^-=2(\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\mathbf{G}_2(t). \tag{87}$$

By applying the Plemelj formula and Cauchy integral formula, we can obtain

$$\mathbf{f}_3(\zeta) = 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left\{\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - a) + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - a)\right\} - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) + \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/\overline{\zeta}_3), \quad |\zeta| < 1, \tag{88a}$$

$$\mathbf{f}_3(\zeta) = 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2\left\{\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/a}{\zeta} + \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/\overline{\zeta}_3) - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a)\right\} + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi} \times \ln(\zeta - 1/\overline{\zeta}_3), \quad |\zeta| > 1. \tag{88b}$$

Substituting Eq. (88b) into Eq. (85), $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ can be expressed as

$$\mathbf{f}_2(\zeta) = (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left\{\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - a) + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - a)\right\} + \sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/a}{\zeta} - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a) + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3). \tag{89}$$

The two expressions of $\mathbf{f}_2(\zeta)$ in its definition region $R < |\zeta| < 1$ obtained from satisfying continuity conditions across the two interfaces must be the same one, then equating Eqs. (83) and (89) yields

$$\mathbf{A}_0 + \sum_{n=1}^{+\infty}\mathbf{A}_n\zeta^n = (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\left\{\mathbf{B}_0 + \sum_{n=1}^{+\infty}\mathbf{B}_n\zeta^n - \frac{(\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_3) + \boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_3)\mathrm{i}}{2\pi}\ln(\zeta - a)\right\} + 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_3\frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3), \tag{90}$$

$$\sum_{n=1}^{+\infty}\overline{\mathbf{B}}_n\zeta^{-n} = (\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\left\{\sum_{n=1}^{+\infty}R^{2n}\overline{\mathbf{A}}_n\zeta^{-n} - \frac{(\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_3) + \boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_3)\mathrm{i}}{2\pi}\ln\frac{\zeta - aR^2}{\zeta}\right\} + 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_1\frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - \zeta_1}{\zeta}. \tag{91}$$

By expanding the above two equations in its definition region and by comparing the coefficients of the save power of ζ, we can obtain

$$(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)\mathbf{A}_n + (\boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_3)\mathbf{B}_n = (\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\frac{(\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_3) + \boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_3)\mathrm{i}}{2\pi}\frac{a^{-n}}{n} - 2\boldsymbol{\Gamma}_3\frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\frac{\zeta_3^{-n}}{n}, \tag{92a}$$

$$(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)\mathbf{B}_n + (\boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_1)R^{2n}\mathbf{A}_n = (\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\frac{(\hat{\mathbf{b}}_1 + \hat{\mathbf{b}}_3) - \boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_3)\mathrm{i}}{2\pi}\frac{(aR^2)^n}{n} - 2\boldsymbol{\Gamma}_1\frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\frac{\overline{\zeta}_1^n}{n}. \tag{92b}$$

From the above equation, we can get

$$\begin{Bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3 & \boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_3 \\ (\boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_1)R^{2n} & \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \mathbf{0}_{2\times2} & (\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2)\frac{a^{-n}}{n} \\ (\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2)\frac{(aR^2)^n}{n} & \mathbf{0}_{2\times2} \end{bmatrix} \begin{Bmatrix} \frac{(\hat{\mathbf{b}}_1+\hat{\mathbf{b}}_3)-\boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1+\hat{\mathbf{f}}_3)\mathrm{i}}{2\pi} \\ \frac{(\hat{\mathbf{b}}_1+\hat{\mathbf{b}}_3)+\boldsymbol{\Gamma}_2^{-1}(\hat{\mathbf{f}}_1+\hat{\mathbf{f}}_3)\mathrm{i}}{2\pi} \end{Bmatrix} + \begin{bmatrix} \mathbf{0}_{2\times2} & -2\boldsymbol{\Gamma}_3\frac{\zeta_3^{-n}}{n} \\ -2\boldsymbol{\Gamma}_1\frac{\bar{\zeta}_1^n}{n} & \mathbf{0}_{2\times2} \end{bmatrix} \begin{Bmatrix} \frac{\hat{\mathbf{b}}_1+\boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi} \\ \frac{\hat{\mathbf{b}}_3-\boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi} \end{Bmatrix} \right\} \quad (n = 1, 2, \ldots, +\infty). \tag{93}$$

The complex potentials which characterize mechanical and electrical fields have been obtained completely as follows:

$$\mathbf{f}_1(\zeta) = 2(\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}\boldsymbol{\Gamma}_2 \left\{ \sum_{n=1}^{+\infty} \mathbf{A}_n\zeta^n - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a) + \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - R^2/\bar{\zeta}_1) - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - 1/a) \right\} + \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - \zeta_1) - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - R^2/\bar{\zeta}_1), \quad |\zeta| < R,$$

$$\begin{aligned}
\mathbf{f}_2(\zeta) &= (\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2)^{-1}(\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2) \left\{ \sum_{n=1}^{+\infty} R^{2n}\bar{\mathbf{A}}_n\zeta^{-n} - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{\zeta - aR^2}{\zeta} + \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - \zeta_1) \right. \\
&\quad \left. - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - aR^2) \right\} + \sum_{n=1}^{+\infty} \mathbf{A}_n\zeta^n - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a) - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - 1/a) \\
&\quad + \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_1^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - \zeta_1) \\
&= (\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_2) \left\{ \sum_{n=1}^{+\infty} \mathbf{B}_n\zeta^n - \frac{\hat{\mathbf{b}}_1 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln(\zeta - a) + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) \right. \\
&\quad \left. - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - a) \right\} + \sum_{n=1}^{+\infty} \bar{\mathbf{B}}_n\zeta^{-n} - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/a}{\zeta} - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a) \\
&\quad + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3), \quad R < |\zeta| < 1,
\end{aligned}$$

$$\mathbf{f}_3(\zeta) = 2(\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_3)^{-1}\boldsymbol{\Gamma}_2 \left\{ \sum_{n=1}^{+\infty} \bar{\mathbf{B}}_n\zeta^{-n} - \frac{\hat{\mathbf{b}}_1 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_1\mathrm{i}}{2\pi}\ln\frac{\zeta - 1/a}{\zeta} + \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/\bar{\zeta}_3) - \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/a) \right\} + \frac{\hat{\mathbf{b}}_3 - \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - \zeta_3) - \frac{\hat{\mathbf{b}}_3 + \boldsymbol{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(\zeta - 1/\bar{\zeta}_3), \quad |\zeta| > 1.$$

Also, the constant terms $\mathbf{A}_0$ and $\mathbf{B}_0$ representing the equipotential field and the translation of a rigid body have been ignored in the above expressions. It can be observed that the above complex potentials indeed satisfy the singularity conditions (17)–(19). We hasten to add that the solution structure for the case when the singularity lies in the inclusion is different from that when it lies in the matrix, so the two cases must be discussed separately. The expressions of stress and electric displacement within the two inclusions and the matrix can be similarly derived from the above complex potentials and will not be shown in this paper.

4. Examples

In this section, several special examples will be presented to demonstrate the versatility and the correctness of the obtained solutions. It is assumed here that the right circular inclusion possesses the same electro-elastic moduli as its surrounding matrix, i.e. $\mathbf{\Gamma}_2 = \mathbf{\Gamma}_1$ and three different kinds of loads will be discussed separately.

4.1. A circular inclusion in piezoelectric matrix subject to remote loadings

It can be found from Eq. (48) that in this case $\mathbf{B}_n = \mathbf{0}$ $(n = 1, 2, \ldots, +\infty)$. Utilizing the mapping function (11), we obtain the field potentials in closed form as follows

$$\mathbf{f}_1(z) = \mathbf{\Pi} z + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2)\overline{\mathbf{\Pi}}\frac{1}{z}, \quad z \in D_1, \tag{94}$$

$$\mathbf{f}_2(z) = \mathbf{\Pi} z + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2)\overline{\mathbf{\Pi}}\frac{1}{z}, \quad z \in D_2, \tag{95}$$

$$\mathbf{f}_3(z) = 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2\mathbf{\Pi} z, \quad z \in D_3. \tag{96}$$

Here, our results coincide with those obtained by Pak (1992) and it can be observed that in this case the stresses and electric displacements are uniform within the circular inhomogeneity.

4.2. A singularity in the matrix

It can be found from Eq. (74) that in this case $\mathbf{B}_n = \mathbf{0}$ $(n = 1, 2, \ldots, +\infty)$. Utilizing the mapping function (11), we obtain the field potentials in closed form as follows:

$$\mathbf{f}_1(z) = \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln(z - z_2) + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2)\frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{z - 1/\bar{z}_2}{z}, \quad z \in D_1, \tag{97}$$

$$\mathbf{f}_2(z) = \frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln(z - z_2) + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2)\frac{\hat{\mathbf{b}}_2 + \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln\frac{z - 1/\bar{z}_2}{z}, \quad z \in D_2, \tag{98}$$

$$\mathbf{f}_3(z) = 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_2\frac{\hat{\mathbf{b}}_2 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_2\mathrm{i}}{2\pi}\ln(z - z_2), \quad z \in D_3. \tag{99}$$

Eqs. (97)–(99) are identical to those obtained by Meguid and Deng (1998).

4.3. A singularity within the left inclusion

It can be found from Eq. (93) that in this case $\mathbf{B}_n = \mathbf{0}$ $(n = 1, 2, \cdots, +\infty)$. Utilizing the mapping function (11), we can obtain the field potentials in closed form as follows:

$$\mathbf{f}_1(z) = \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln z + 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_3\frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{z - z_3}{z}, \quad z \in D_1, \tag{100}$$

$$\mathbf{f}_2(z) = \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_2^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln z + 2(\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}\mathbf{\Gamma}_3\frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln\frac{z - z_3}{z}, \quad z \in D_2, \tag{101}$$

$$\mathbf{f}_3(z) = \frac{\hat{\mathbf{b}}_3 - \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(z - z_3) + (\mathbf{\Gamma}_2 + \mathbf{\Gamma}_3)^{-1}(\mathbf{\Gamma}_2 - \mathbf{\Gamma}_3)\frac{\hat{\mathbf{b}}_3 + \mathbf{\Gamma}_3^{-1}\hat{\mathbf{f}}_3\mathrm{i}}{2\pi}\ln(z - 1/\bar{z}_3), \quad z \in D_3. \tag{102}$$

Eqs. (100)–(102) are identical to those obtained by Deng and Meguid (1999).

5. Numerical demonstration

As one example, consider now the case when the matrix is subjected to uniform remote electro-mechanical loadings.

The material properties of the matrix are
$c_{44} = 3.53 \times 10^{10}\ \mathrm{N\,m^{-2}}$, $e_{15} = 10\ \mathrm{C\,m^{-2}}$, $\varepsilon_{11} = 1.51 \times 10^{-8}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$.

For simplicity, the material properties of the left and right circular inclusions are
$c_{44} = 1.00 \times 10^{9}\ \mathrm{N\,m^{-2}}$, $e_{15} = 20\ \mathrm{C\,m^{-2}}$, $\varepsilon_{11} = 1.51 \times 10^{-8}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$.

Assume the two inclusions approach each other very closely with the geometry parameters taken as $x_2 = 1.01$ m and $x_1 = 4.01$ m.

The remote loading conditions are $\sigma_{zx}^{\infty} = 2.5 \times 10^7\ \mathrm{N\,m^2}$, $\sigma_{zy}^{\infty} = 5 \times 10^7\ \mathrm{N\,m^{-2}}$; $E_x^{\infty} = 0.5 \times 10^6\ \mathrm{V\,m}$, and $E_y^{\infty} = 10^6\ \mathrm{V\,m}$.

In the calculation, Laurent's series is truncated at $n = 200$ due to the fact that the two inclusions are extremely close (In general situations, not more than 100 terms in Laurent's series are needed to exactly satisfy all of the prescribed boundary conditions). All of the calculated quantities have been divided by 10^7.

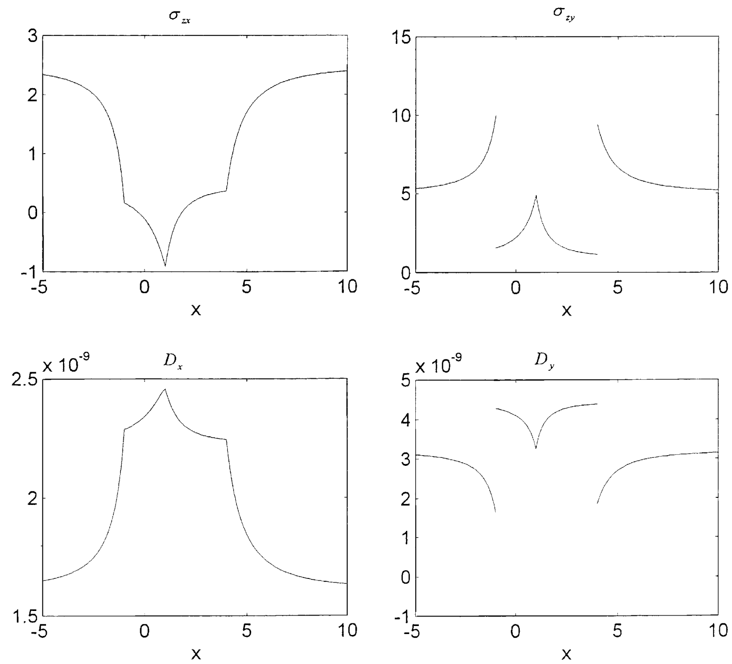

Fig. 2. Distribution of stress and electric displacement along the x-axis.

Fig. 2 shows the distributions of the stresses and electric displacement along the x-axis. From this figure, it can be observed that σ_{zx} and D_x are continuous across the interface boundaries while σ_{zy} and D_y are discontinuous across the boundaries. We also note that the stress and electric displacement are no longer uniform within the two inclusions. Fig. 3 shows the angular variations of stresses and electric displacement along the interface formed between the left inclusion and the matrix, whereas Fig. 4 shows the angular variations of stresses and electric displacement along the interface formed between the right inclusion and the matrix. It can be observed from Figs. 3 and 4 that the normal components of stress and electric displacement are continuous across the two interfaces, while their tangential components have a jump across the two interfaces, which conforms to the prescribed boundary conditions. From these curves, we can also find that significant stress amplifications will occur along these interfaces. Fig. 5 illustrates the case when the right inclusion and the matrix have the same electro-elastic moduli. We can find that in this situation the stress and electric displacement are uniform within the left inclusion, which confirms our solution from one aspect. Fig. 6 shows the case when the right inclusion is far away from the left inclusion. In this case, the stresses and electric displacement in both of the two inclusions are uniform, which also confirm our solutions. To see more clearly the distribution of stresses in the full field, Figs. 7 and 8 present the contour plots of $\sigma_{zy}/\sigma_{zy}^{\infty}$ and $\sigma_{zx}/\sigma_{zy}^{\infty}$ when subjected to uniform loads $\sigma_{zx}^{\infty} = 0$, $\sigma_{zy}^{\infty} = 5 \times 10^7\ \mathrm{N\,m^{-2}}$, $E_x^{\infty} = 0$, $E_y^{\infty} = 10^6\ \mathrm{V\,m^{-1}}$. It can be observed that σ_{zy} is symmetric with respect to the x- axis, whereas σ_{zx} is antisymmetric with respect to the x-axis and that there exists serious stress amplification at the point where the two inclusions are nearly in contact with each other, also we can see clearly from the two figures that how the stress fields within each of the two inclusions are altered by the existence of another circular inclusion.

We have carried out a large quantity of calculations and find that the solution provided in this paper is especially suited to treat the case when the two circular inclusions are extremely closely spaced provided that the number of terms in Laurent's series is sufficiently large.

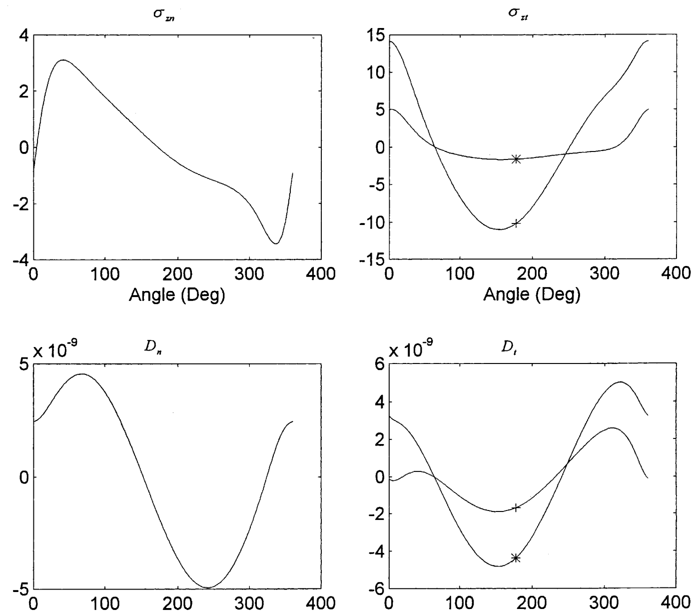

Fig. 3. Angular variations of stress and electric displacement along the interface formed between the left inclusion and the matrix ('$*$' for inclusion, '+' for matrix).

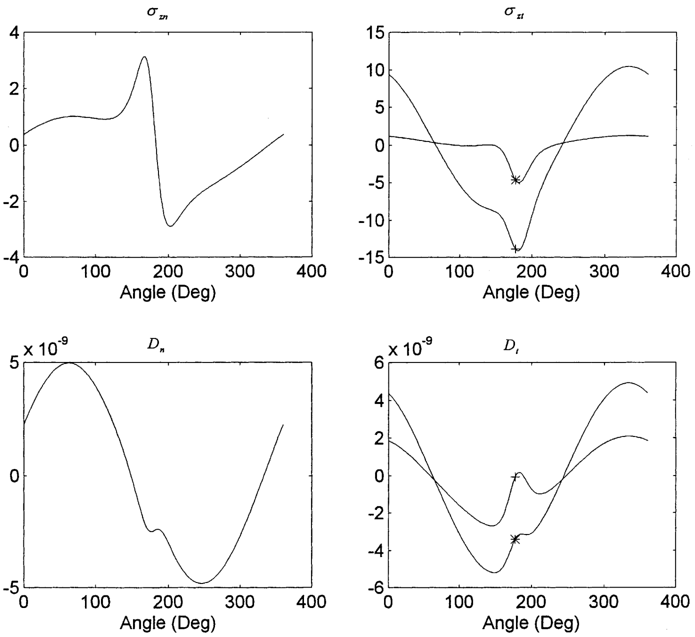

Fig. 4. Angular variations of stress and electric displacement along the interface formed between the right inclusion and the matrix ('$*$' for inclusion, '+' for matrix).

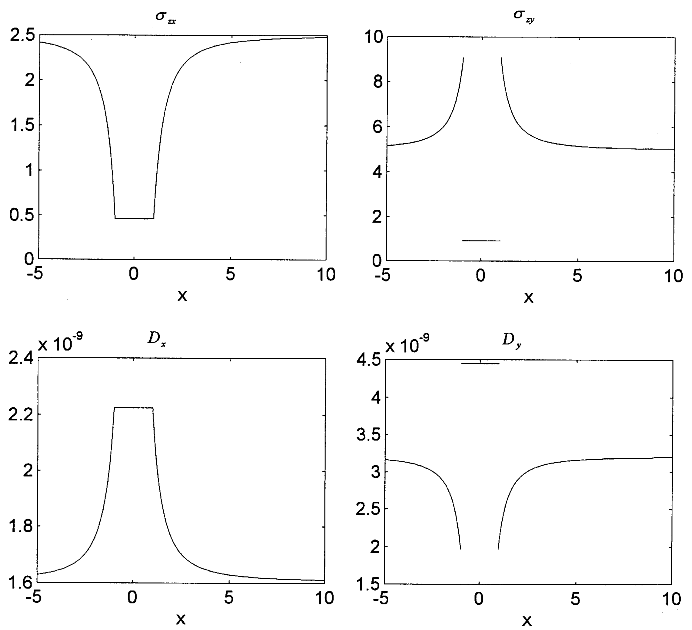

Fig. 5. Distribution of stress and electric displacement along the x-axis for the case when the right inclusion and the matrix have the same material property.

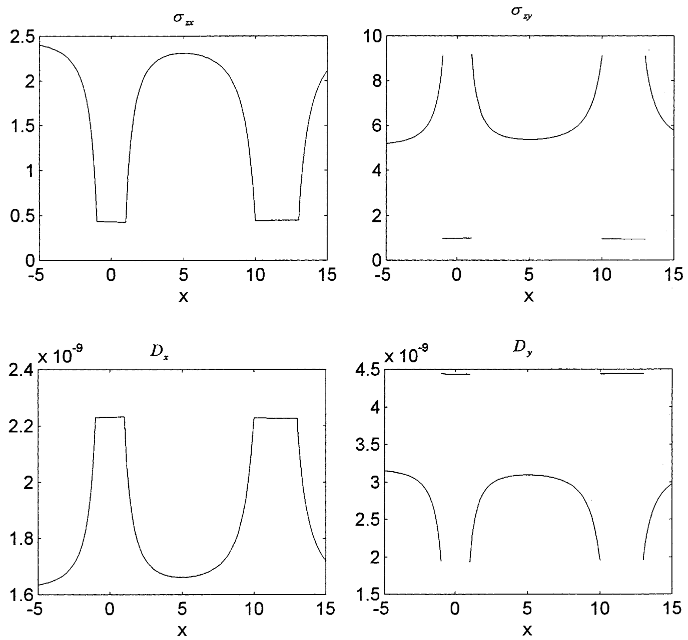

Fig. 6. Distribution of stress and electric displacement along the x-axis for the case when the right inclusion are far away from the left inclusion.

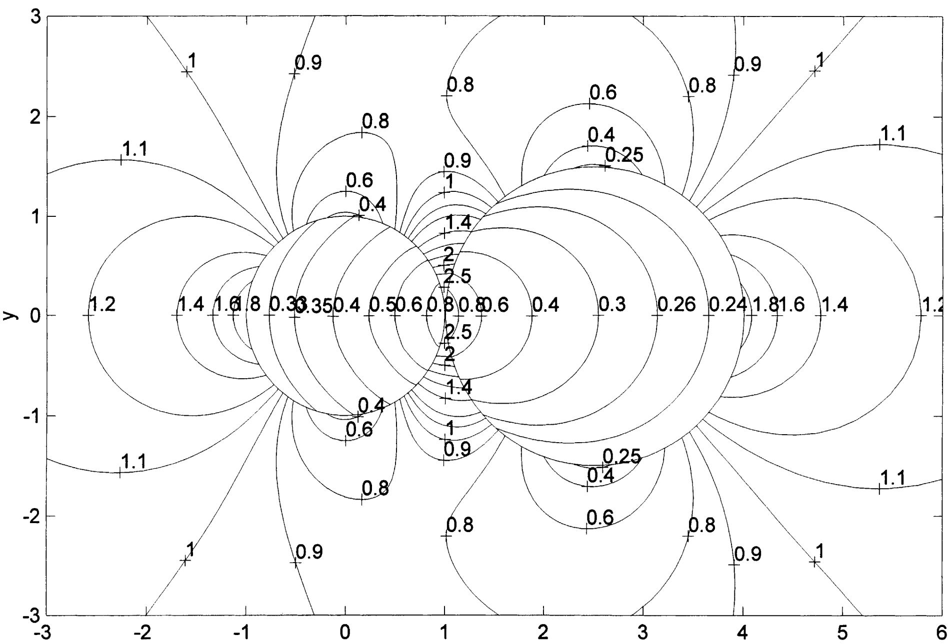

Fig. 7. Contour plots of $\sigma_{zy}/\sigma_{zy}^{\infty}$ when subjected to uniform loads: $\sigma_{zx}^{\infty} = 0$, $\sigma_{zy}^{\infty} = 5 \times 10^{7}$ $\mathrm{N\,m^{-2}}$, $E_x^{\infty} = 0$, $E_y^{\infty} = 10^{6}$ $\mathrm{V\,m^{-1}}$.

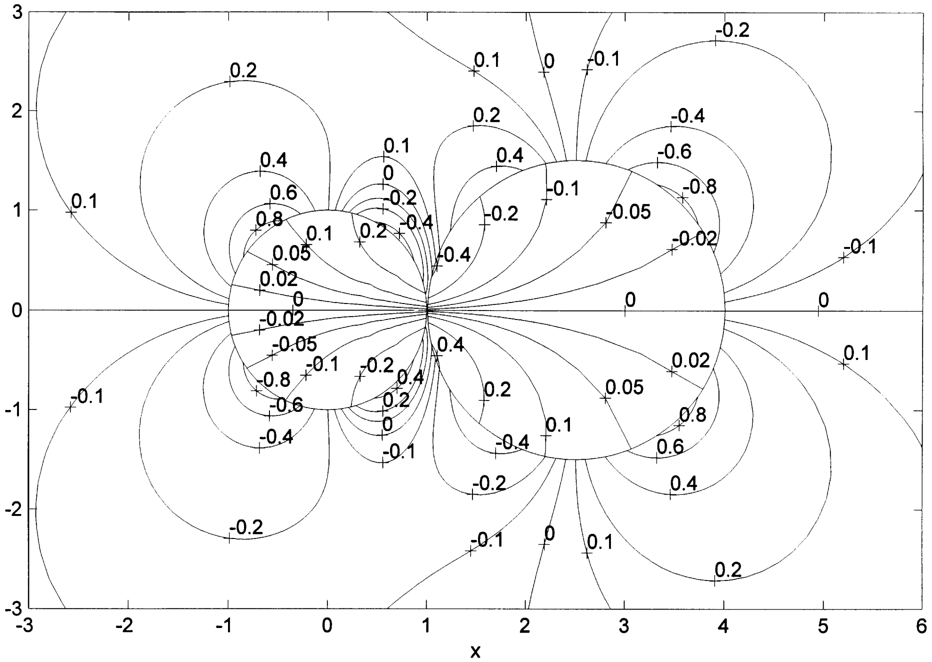

Fig. 8. Contour plots of $\sigma_{zx}/\sigma_{zy}^{\infty}$ when subjected to uniform loads $\sigma_{zx}^{\infty} = 0$, $\sigma_{zy}^{\infty} = 5 \times 10^7\ \mathrm{N\,m^{-2}}$, $E_x^{\infty} = 0$, $E_y^{\infty} = 10^6\ \mathrm{V\,m^{-1}}$.

6. Conclusion

An analytical solution in series form for the problem of double circular piezoelectric inclusions embedded in an infinite piezoelectric matrix is derived by applying the complex variable method. The generality of the solution is shown in the following sense: (1) The size, location and electro-elastic properties of the two circular inclusions are arbitrary, (2) The loadings (singularities) are arbitrary. From the obtained result, it can be shown that the interaction effect cannot be ignored when the inclusions are closely spaced. The obtained basic solution provided in this paper can be used as a Kernel function of a singular integral equation to consider the interactions among the two circular inclusions and cracks or rigid line inclusions (anticracks). We will present the result in a forthcoming paper.

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